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NON-SMOOTH MECHANICAL SYSTEMS†

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A review of research on non-smooth dynamical systems from an applications point of view is given. Two types of non-smoothness are considered: caused by impacts or by friction. For each type a list of basic models is given, areas of application are specified, and the results of research are reviewed. © 2001 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

By providing some examples of practical relevance, we hope to stimulate further work in non-smooth mechanics, since they now represent a challenge to both engineers and mathematicians. In the examples presented below, non-smooth phenomena are caused by kinematic constraints or physical effects like friction, impacts or backlash. In the modelling of real practical problems such phenomena have been considered as errors and have therefore been neglected for a long time. Later, these phenomena were considered in an approximate manner by smoothed characteristics often unjustified from the physical paint of view. Just recently, in refined and more precise models, these phenomena have been taken into account correctly as non-smooth effects. However, building adequate models for real practical problems depends on our knowledge of the processes involved and also on the nature of the problem itself. Hence, in applications the golden rule still holds, namely, models must be as simple as possible and as accurate as necessary.

This means that physical effects like friction and impacts are usually modelled in a simple way and described, e.g. by Coulomb's friction law and Newton's impact law leading to a non-smooth force and motion characteristic, respectively. Gaining better insight may result in an exchange of these non-smooth models by smooth ones. This verifies that modelling is an interesting and non-trivial process, sometimes called an "engineering art". However, from a pragmatic application point of view the question remains: How can we deal with non-smooth dynamical systems?

Simply speaking, a non-smooth system is characterized by force and/or motion characteristics which are not continuous or non-differentiable. Figure 1 shows three simple mechanical force elements (left) with non-smooth force characteristics (right). Usually these elements are coupled to masses and other elements to form a dynamical system. In case 1(a) we have a system of linear springs (with stiffness k), with a backlash of magnitude $2a$. The force characteristic has the form

$$f_F = \begin{cases} k(s-a) & \text{if} \quad s \geqslant a \\ 0 & \text{if} \quad |s| \leqslant a \\ k(s+a) & \text{if} \quad s \leqslant -a \end{cases}$$

In cases 1(b) and 1(c) the corresponding forces change their sign if the displacement $\dot{s}$ and the velocity s change sign, respectively. Figure 1(b) shows a system of springs (with stiffness k) with preload $|f_0| = ka$ and force characteristic

$$\begin{aligned} f_F &= k(s+a) && \text{if} \quad s > 0 \\ |f_F| &\leqslant ka && \text{if} \quad s = 0 \\ f_F &= k(s-a) && \text{if} \quad s < 0 \end{aligned} \tag{1.1}$$

Figure 1(c) represents a pair of dry Coulomb friction with coefficient μ; here f_N is the normal load. In this case the force characteristic is

$$\begin{aligned} f_R &= \mu f_N && \text{if} \quad \dot{s} > 0 \\ |f_R| &\leqslant \mu f_N && \text{if} \quad \dot{s} = 0 \\ f_R &= -\mu f_N && \text{if} \quad \dot{s} < 0 \end{aligned} \tag{1.2}$$

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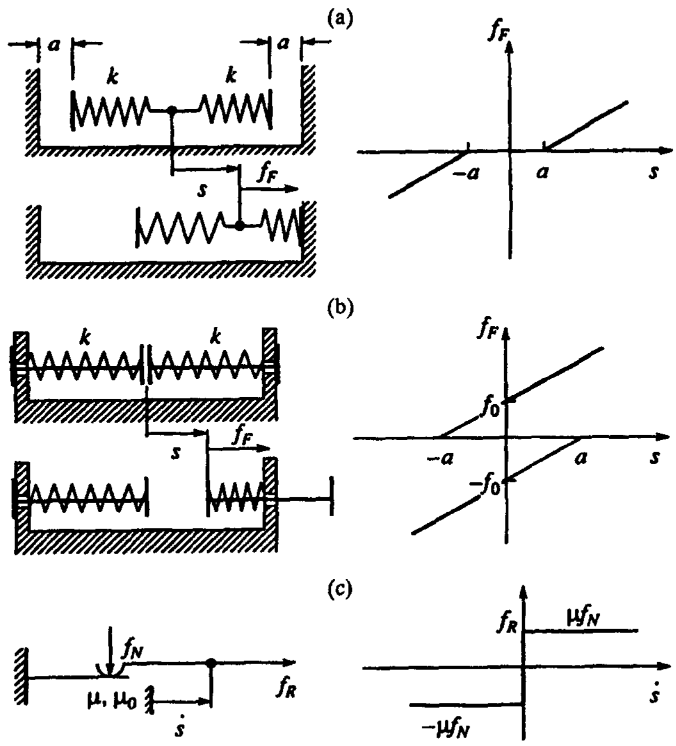

Fig. 1

The vertical part of the force characteristics (1.1) and (1.2) near the origin has a special mechanical meaning, here the force is not an applied force but a reaction or constraint force which cannot be determined from the force characteristic. However, there are enough reaction or constraint equations in a mechanical system available to calculate these forces uniquely. This is an example of the general fact, that, if we prescribe the displacements or velocities in a system, i.e. if we improve kinematic constraints on it, we get reaction or constraint forces and at the same time we may change the number of degrees of freedom. Figure 2 illustrates this by three simple motion characteristics

In case 2*a* a point mass moves freely in the region $s \leqslant 0$ and collides elastically with a rigid wall with $s = 0$; here $\dot{s}_+ = -e\dot{s}_-$ where $0 < e \leqslant 1$. In case 2*b* the system consists of two point masses with coordinates s_1 and s_2; the unilateral constraint is described by the inequality $s_2 - s_1 \geqslant 0$. When $s_1 = s_2$, ideally plastic impact occurs ($e = 0$) and the number of degrees of freedom is changed from two to one by the impact. The post-impact velocities are expressed by the formula

$$\dot{s}_{1+} = \dot{s}_{2+} = \frac{m_1\dot{s}_{1-} + m_2\dot{s}_{2-}}{m_1 + m_2}$$

A similar situation occurs in systems if certain positions are fixed, e.g. in snapper mechanisms. An example is the deployment of folded solar panels from a spacecraft. When contiguous panels have reached their final position their relative motion is suddenly stopped, resulting in a reduction in the number f of degrees of freedoms. Another example is a point mass suspended on a thread of length l in a plane (case 2*c*), where the position of the point mass at the end of the thread can lie within a circle of radius $l (r < l, f = 2)$ or on its boundary ($r = l, f = 1$). Here, there is a unilateral constraint of the form $g(x_i, \dot{x}_i, t) \leqslant 0$, where g denotes a function which depends on kinematic quantities, (see, e.g. Fig. 2(c) where $g = x_1^2 + x_2^2 - l^2$)

Alternatively, this unilateral constraint can also be formulated using the tension force f_N of the thread. For a mass position within the circle the thread is not in tension and $f_N = 0$ when $g < 0 (f = 2)$. However, when the mass is positioned on the boundary of the circle the thread is in tension so that $f_N > 0$ for $g = 0$ ($f = 1$). Thus, for the tension force we have $f_N \geqslant 0$, which supplements the constraint equation $g \leqslant 0$. Both conditions can be combined using the complementary relation $gf_N = 0$ which holds at anywhere t. Hence, as a complete description of the unilateral constraint we get

$$g \leqslant 0, \quad f_N \geqslant 0, \quad gf_N = 0$$

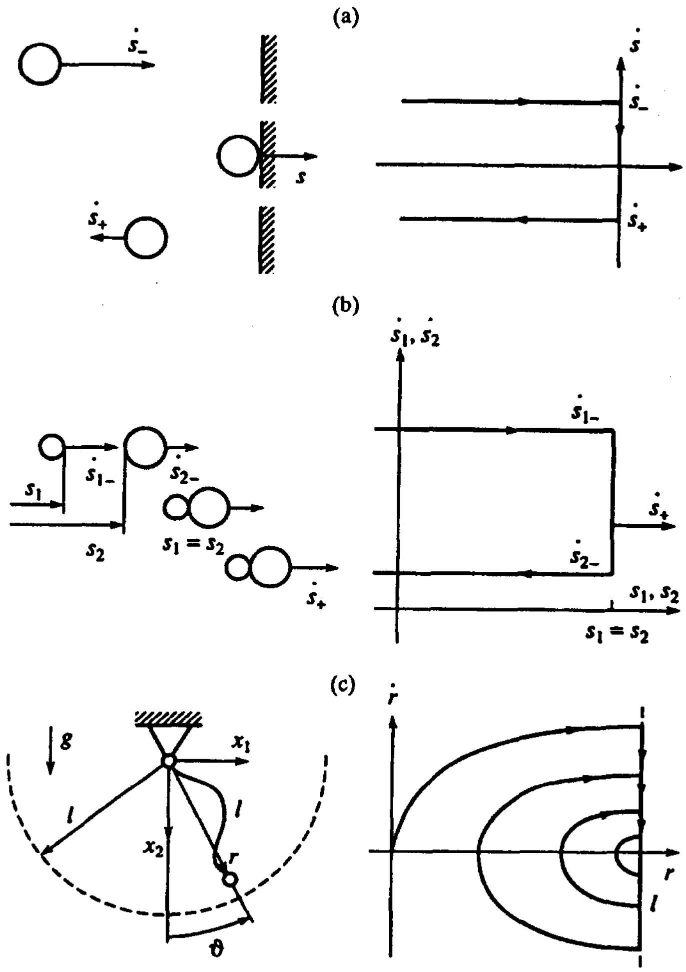

Fig. 2

This formula has plays the central role in all contact problems with unilateral constraints.

Other examples of non-smooth systems can be found in the area of active control, where constraints are usually imposed on the control variables. Thus, in closed-loop control some switching takes place depending on the states of the system when the constraints are active, e.g. time optimal control or bang-bang control [1]. Other examples are systems with variable structure, where switching takes place from one structure to another, finally reaching the control goal by a so-called sliding mode, cf. [2]. In the following we also use the term, "switching" in passive systems in order to have a general formulation.

The mathematical description of non-smooth systems consists of three parts:

(i) a smooth mathematical description for each switching state of the system,

(ii) an indicator function or switching surface defining where a non-smooth transition in state space is given,

(iii) a transition function defining how the change from one switching state to another occurs.

If we consider two switching states p and q, described by ordinary differential equations in state space, we get the following mathematical description:

(i) prior to switching ($t < t^*$)

$$\underline{\dot{z}} = \underline{f}_p(\underline{z}),\ \underline{z}(t_o) = \underline{z}_o,\ \underline{f}_p \in C^1$$

after switching ($t > t^*$),

$$\underline{\dot{z}} = \underline{f}_q(\underline{z}),\ \underline{z}(t_+^*) = \underline{z}_+,\ \underline{f}_q \in C^1$$

(ii) switching surface S_p

$$S_p = \{\underline{z} \in R^n \mid \underline{h}_p(\underline{z}) = \underline{0}\}$$

(iii) transition function for $t = t^*$ and $\underline{h}_p(\underline{z}) = 0$

$$\underline{z}(t_+^*) = u_{pq}[\underline{z}(t_+^*)]$$

Let us take the impact of a point mass on a rigid wall as an example, (Fig. 2, case *a*). The state variables are s and $\dot{s}$. Let us assume that the velocities of the point mass before and after the impact are constant. Then, the equations corresponding to (i)–(iii) are:

(i) prior to switching ($t < t^*$)

$$\dot{s}_- = \dot{s} = const, s = \dot{s}_- t + s_o$$

after switching ($t > t^*$)

$$\dot{s} = \dot{s}_+ = const, s = \dot{s}_+ t + s_+$$

(ii) switching surface

$$s = \{s \mid s = 0\}$$

(iii) transition function for $t = t^*$ and $s = 0$

$$\dot{s}_+ = -e\dot{s}_-, \qquad s_+ = s_- = 0$$

Below a review will be given of non-smooth systems. Since we are dealing with complex non-linear dynamical problems, chaotic motions will also appear. The existing literature will be reviewed together with applications; there is also some general literature on non-smooth systems available. We mention, in particular the course material [3], the IUTAM conference proceedings [4], lecture notes [5], and the monograph [6], where multiple impact problems are also considered (which are omitted here) and, furthermore, the monographs [7, 8].

2. EXAMPLES OF NON-SMOOTH SYSTEMS

The following examples are far from complete; this is also true for the references. The problem arises of how to arrange the examples, in particular, if more than one non-smooth physical effect is involved. Here, the division into major effects has been chosen, i.e. problems with impacts and problems with friction. Simple models are mentioned first, then more complicated applications.

Problems with impacts. Figure 3 shows a collection of simple impact problems together with the corresponding mechanical models – which can easily be built for demonstration. Figure 3(a) is a classroom example where the phase portrait consists of two families of parabolas with a non-smooth transition [9]. The rocking block problem, Fig. 3(b), dates back to 1956, [10, 11], where the design of foundations for buildings subject to earthquakes has been discussed. In [12] small-scale experiments using a vibrating table with harmonic and random excitation were performed. In [13], in particular, the impacts of free rocking blocks were investigated. The forced response has been investigated, e.g. in [14–17], as well as in [18–20], where impacts on the sidewalls of the block were also investigated.

The bouncing ball problem (Fig. 3*c*), was investigated in [21, 22] for random repeated impacts. Impacts due to a harmonically vibrating table were considered in [23, 24] and, under some simplifying assumptions, a two-dimensional discrete mapping was found. The boundaries of the region of attraction were calculated in [25], and were found to be fractal curves. The regular motion behaviour is governed by the number of excitation periods between subsequent impacts and the periodicity of the motion itself. This behaviour is similar to the operation of a percussion drilling machine, [26].

Another problem of practical importance are rattling gear boxes when the gear-wheels are not loaded. A simple mechanical model is shown in Fig. 3(d), where impacts occur in due to backlash, [27]. This problem has been studied extensively in [28–31]. Periodic and chaotic motions have been found. The results of a simulation have been verified by experiments. A good review is given in [32], where further references can be found. Similar investigations are described in [33, 34]. A related but different impulsive process is hammering, where the gear-wheels are loaded. Results are given in [32, 35, 36]. Further applications are impacts of heat exchanger tubes [37, 38] and clearance connections in high-speed machinery [39, 40].

Figure 3(e) shows a classical impact oscillator. It has been investigated in [41] under the name of a vibro-impact system. An extensive analysis has been given in [42–44], where chaotic motions are included. The transition to chaos has also been investigated in [45]. Technical applications are impact print hammers [46–48] and pile-drivers [49]. A new type of bifurcation has been found, namely grazing bifurcation, where the limiting case of an impact

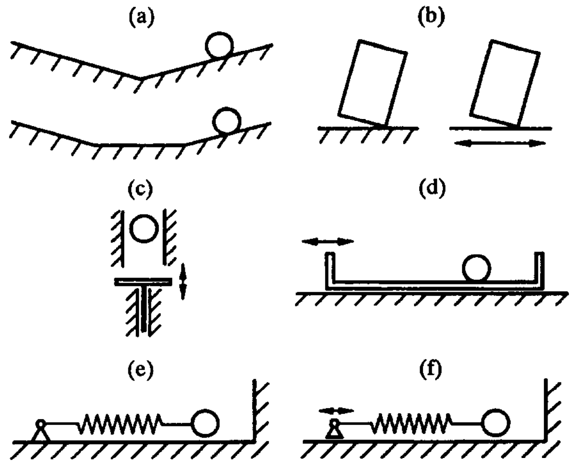

Fig. 3

with zero velocity is given [50, 51]. New numerical and experimental results are presented in [52–54], where further references can be found.

So far, only free and forced vibrations have been mentioned. However, there are also non-smooth parametric vibrations as well as self-excited vibrations. Impulsive parametric excitation problems using the point mapping approach, among other methods, were investigated in [55, 56]. A review and particular applications to vehicles on elevated guideways are given in [57, 58], resulting in systems with jumping state variables.

With regard to self-excited vibrations, the classical problem of a clock pendulum, the electric bell and a variety of hydraulic self-excited oscillatory systems can be found in [9]. The "woodpecker", a mechanical toy which exhibits self-excited vibrations has been analysed in [59, 60]. An important technical system, the railway bogie, has been investigated extensively in [59–61]. Here also chaotic motions have been found.

Problems with friction. Dry friction appears in two different phenomena in nature:

1. as a resistance against the beginning of a motion from equilibrium (the sticking mode), where the friction force is a constraint force,
2. as a resistance against an existing motion (the slip mode), where the friction force is an applied force.

In vibration problems with friction, summarized in Fig. 4, both phenomena occur. In a simple friction oscillator, Fig. 4(a), friction leads to energy dissipation until the mass sticks in a so called dead zone [9]. For the oscillators Fig. 4(b)–(e) both phenomena can occur successively, resulting in stick-slip motions. Here, a rich literature is available describing the state-of-the-art [64–67]. The physics of friction is addressed in [68, 69]. A review of structural vibrations with friction contacts has been given in [70] and will not be repeated here. Some additional references will be given below for the problems summarized in Fig. 4.

Figure 4(b) shows a friction oscillator with external excitation dating back to 1931 [71]. Harmonic or random excitation has been investigated in [72–74].

Friction oscillators with pure self excitation, Fig. 4(c), result in robust limit cycles. The energy is transferred from the moving belt to the oscillator either by friction forces with a decreasing characteristic [9, 67, 75], or, in the case of constant friction forces, by fluctuating normal forces [76, 77]. The resulting stick-slip vibrations appear in everyday life as well as in engineering systems. Example are the sound of bowed instruments, singing wine glasses, creaking doors, squeaking chalks, and also rattling joints of robots, chattering machine tools, grating brakes and the squealing noise of tramways in narrow curves. The squealing noise of railway wheels has been investigated in [78, 79]. Investigations of machine tool chatter can be found in [80,81].

It is possible to break up the robust limit cycle by using an additional external excitation, (see Fig. 4d). This case has been investigated extensively in [53, 82–89]. Here, a rich bifurcational behaviour has been found which were analysed using numerical simulations and the mapping approach; experiments have also been performed. An analysis of some multi-degree-of-freedom friction oscillators, Fig. 4(e), can be found in [90–92] as well as in [54].

3. CONCLUDING REMARKS

It is not appropriate here, to draw any conclusions. Instead, we will use some questions.

1. How can non-smooth systems be solved efficiently? Solution methods like piecewise solutions, the mapping approach, smoothing techniques and numerical simulations have to be evaluated and improved.
2. How can the properties of the solutions of non-smooth systems be analysed? Efficient methods for stability and bifurcation analysis are required.

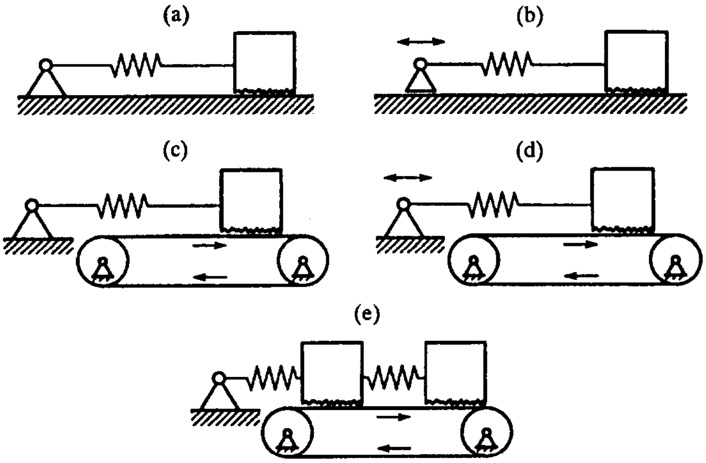

Fig. 4

3. How can the attractor characteristics for non-smooth systems be reconstructed? Signal-and model-based methods for computing Lyapunov exponents and the dimension of attractors need to be developed. All these methods must be suitable for analysing non-smooth multi-degree-of-freedom systems which occur in applications.

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